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ELASTOSTATIC SOLUTION FOR A CIRCULAR MEMBRANE BONDED TO A HALF-SPACE UNDER PLANE LOADING†

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Abstract—The solution is given for the elastostatic load transfer problem of a circular elastic membrane bonded to the boundary of a materially dissimilar half-space, which is loaded far from the membrane by a state of plane stress parallel to its boundary.

The problem is reduced to a system of Fredholm integral equations of the first kind, with logarithmic singularities in the kernals, for the unknown bond force between the membrane and the half-space. These integral equations are solved exactly and the solution of the problem is obtained in the form of elementary functions for the limiting case of an inextensible membrane. For the elastic membrane the integral equations are handled directly by numerical techniques and the perturbation of the uniform stress field due to the attached membrane is obtained numerically for several combinations of the material and geometrical parameters.

A certain ratio of the membrane to half-space strain is computed which gives an indication of the accuracy of strain measurements obtained in experimental stress analysis by the use of bonded gages. These results indicate that the "average strain" in the membrane is close to the far-field half-space strain only when $\mu a/\hat{\mu}h > 100$, where $\hat{\mu}$, h, a are the shear modulus, thickness and radius of the membrane and μ is the shear modulus of the half-space.

1. INTRODUCTION

THE problem of a thin circular disk attached to a half-space under plane biaxial loading at infinity is solved within the theory of linear, isotropic and homogeneous elastostatics. The disk is assumed to have no bending stiffness and is treated as a two-dimensional continuum in the sense of generalized plane stress (i.e. as an elastic membrane). It is assumed to be perfectly bonded to the half-space, which has elastic constants different from those of the membrane.

A complete discussion of the numerous load-transfer problems solved previously is beyond our scope. A general lecture and accompanying paper by Sternberg [1] gives a thorough discussion of several load transfer problems recently solved by Sternberg and Muki. This paper includes numerous references through which many of the important historical developments on load-transfer problems can be traced. The works by Bufler [2], who treats the problem of a finite elastic bar bonded to the edge of an elastic half-plane, and by Aleksandrov and Solov'ev [3], who consider an inextensible elliptical membrane bonded to an elastic half-space, are more closely related to the problem considered here.

† Based on the doctoral dissertation of M.A. Hamstad in partial fulfillment of the requirements for the Ph.D. degree in Engineering, University of California, Berkeley.

One important contribution of the work outlined in [1] is the rational treatment there of a phenomenon, which was observed previously by Reissner [4] in connection with load transfer from a bar to a sheet and by Goodier and Hsu [5] for two overlapped sheets, that concentrated forces may be transmitted between the bonded elastic members at the boundary of their contact region. The possibility that such concentrated loads may occur should not be precluded in the formulation of the problem, and this question need not be settled on the basis of physical arguments. Instead, it should be resolved by an analysis of the integral equations governing the load transfer functions.

In addition to its theoretical interest, the elasticity problem worked here has practical application especially in the field of experimental stress analysis. When a strain gage is bonded or a strain sensitive coating is applied to the surface of an elastic body, the state of stress in the vicinity of the attachment is altered. The extent of this reinforcement clearly depends on the relative stiffness of the body and the attachment. When the stiffness of the gage or coating is relatively small, the disturbance of the stress field is also small and can be accounted for by the use of standard calibration experiments. But if the attachment has considerable stiffness relative to the elastic body, its local effect can be quite significant, and what is more important, the effect can vary widely with the experiment so that a calibration based on a standard test cannot be used for different experiments. Faced with this problem it is important to know how the reinforcement effect depends on the relative stiffness and how this effect decays with distance from the attachment.

The methods used in treating the problem stated in the first paragraph will be briefly outlined. First, the problem is formulated mathematically and the general plane loading at infinity is decomposed into two cases: isotropic stress and pure shear. Then auxiliary membrane and half-space solutions are written in terms of the unknown transfer functions, which include the possibility of concentrated load transfer at the edge of the membrane. The integral equations governing the load transfer functions are then derived by the use of these solutions with the bond conditions. These integral equations are first examined in order to determine the concentrated edge loads. Then for the special case of an inextensible membrane the integral equations are solved exactly, and the stress and displacement fields are determined in the half-space in terms of elementary functions. For the elastic membrane under isotropic loading the singular integral equation of the first kind is solved directly by numerical methods for various values of the material and geometrical parameters. This numerical solution is then used in the numerical computation of representative stress components in the membrane and the half-space. These latter results are illustrated graphically to demonstrate how the effect of the membrane on the stress field in the half-space decays with distance from the membrane. A ratio of an average membrane strain to the far field half-space strain and its dependence on the parameters is also illustrated graphically.

2. FORMULATION OF THE PROBLEM

Let R and $\hat{R}$ denote the open half-space and the open disk of radius a, and let π and $\hat{\pi}$ represent their boundaries (see Fig. 1). Then

$$
\begin{aligned}
R &= \{\mathbf{x}|x_3 > 0\}, \qquad & \hat{R} &= \{\mathbf{x}|x_1^2+x_2^2 < a^2, x_3 = 0\}, \\
\pi &= \{\mathbf{x}|x_3 = 0\}, \qquad & \hat{\pi} &= \{\mathbf{x}|x_1^2+x_2^2 = a^2, x_3 = 0\}.
\end{aligned}
\tag{2.1}
$$

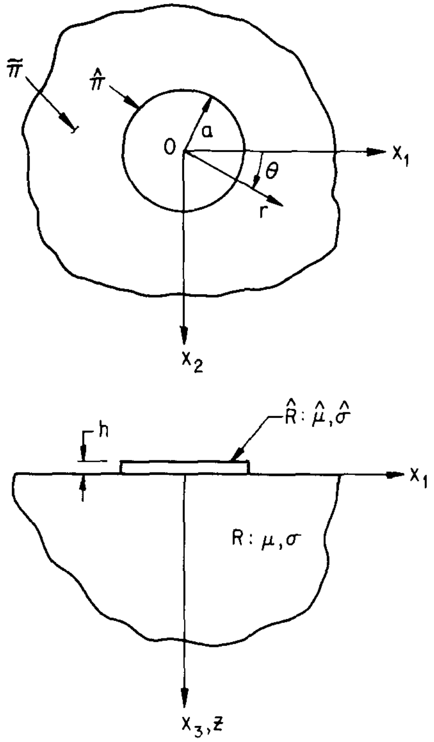

FIG. 1. Half-space and attached membrane.

It is also convenient to define the set $\tilde{\pi}$ as

$$\tilde{\pi} = \{\mathbf{x} | x_1^2 + x_2^2 > a^2, x_3 = 0\}, \tag{2.2}$$

so that

$$\pi = \hat{R} \cup \hat{\pi} \cup \tilde{\pi}. \tag{2.3}$$

Assume that R is occupied by an isotropic, homogeneous and elastic solid characterized by shear modulus μ, and Poissons ratio σ, while $\hat{R}$ is the cross-section of the thin cylindrical domain of height h with elastic constants $\hat{\mu}$ and $\hat{\sigma}$. Let $\{u_i, \sigma_{ij}, \varepsilon_{ij}\}$ denote the three-dimensional displacement, stress and strain fields on R and $\{\hat{u}_\alpha, \hat{\sigma}_{\alpha\beta}, \hat{\varepsilon}_{\alpha\beta}\}$ denote the corresponding two-dimensional quantities on $\hat{R}$, which are to be interpreted as the thickness averages of these fields on the thin cylindrical domain in the sense of generalized plane stress. Then the field equations of linear elastostatics appear in cartesian form as†

$$\sigma_{ij,j} = 0, \qquad \sigma_{ij} = 2\mu\left[\left(\frac{\sigma}{1-2\sigma}\right)\varepsilon_{kk}\delta_{ij} + \varepsilon_{ij}\right], \tag{2.4}$$
$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}),$$

† Summation is implied by repeated indices. The range of Latin indices is (1, 2, 3) and that of Greek indices is (1, 2).

on R, and

$$\hat{\sigma}_{\alpha\beta,\beta}+\hat{f}_{\alpha}=0, \qquad \hat{\sigma}_{\alpha\beta}=2\hat{\mu}\left[\left(\frac{\hat{\sigma}}{1-\hat{\sigma}}\right)\hat{\varepsilon}_{\gamma\gamma}\delta_{\alpha\beta}+\hat{\varepsilon}_{\alpha\beta}\right], \tag{2.5}$$
$$\hat{\varepsilon}_{\alpha\beta}=\tfrac{1}{2}(\hat{u}_{\alpha,\beta}+\hat{u}_{\beta,\alpha}),$$

on $\hat{R}$, where $\hat{f}_{\alpha}$ denote the body forces for the membrane which arise as a result of the bond condition as will be seen later on.

The boundary conditions, bond conditions and regularity conditions are

$$\left.\begin{array}{l} h\hat{f}_r+h\hat{\sigma}_{rr|r=a}\delta(r-a)-\sigma_{zr}=0, \\ h\hat{f}_\theta+h\hat{\sigma}_{r\theta|r=a}\delta(r-a)-\sigma_{z\theta}=0, \qquad \sigma_{zz}=0, \end{array}\right\} \text{ on } \pi \tag{2.6}$$

where

$$\hat{f}_r=\hat{f}_\theta=0 \quad \text{on } \tilde{\pi}, \tag{2.7}$$

and

$$\hat{u}_r=u_r, \hat{u}_\theta=u_\theta, \hat{u}_z=u_z \quad \text{on } \hat{R}, \tag{2.8}$$

$$\sigma_{11}\to\sigma_{\mathrm{I}}, \sigma_{22}\to\sigma_{\mathrm{II}}, \sigma_{12}\to 0, \qquad \sigma_{3i}\to 0 \quad \text{as} \quad |\mathbf{x}|\to\infty \text{ in } R. \tag{2.9}$$

Equations (2.6), (2.7) imply the conditions of vanishing tractions on the half-space outside the area of contact as well as the equilibrium of the force transmitted between the membrane and half-space over the contact area $\hat{R}$. The appearance of the delta function in (2.6) reflects the possibility that the half-space and membrane may transmit a concentrated ring-load to each other along the ring $r = a$.† The $\hat{f}_r$ and $\hat{f}_\theta$ appearing in equations (2.5)–(2.7) arise from the unknown load transfer forces between the membrane and the half-space. These quantities appear as body forces in the two-dimensional model of the membrane. Due to the fact that the membrane has no bending stiffness, no force can be generated at $\hat{R}$ by the assembly in the z-direction, and $\hat{u}_z$ can assume any value. Hence, the condition $\hat{u}_z = u_z$ in (2.8) places no displacement boundary condition on the half-space, and therefore the bond condition in this direction merely becomes the normal traction condition on the half-space. Equation (2.9) expresses the fact that the loading on the half-space is a plane biaxial stress field far from the membrane. The coordinate axes have been chosen to coincide with the principle directions of this plane stress field, so that σ_{I} and σ_{II} are the prescribed principle stresses.

In order to solve the boundary value problem governed by equations (2.4)–(2.9), it is convenient to first consider two special cases of the loading which can then be appropriately combined (due to the linearity of the governing equations) to produce the required solution. Thus let S be the required solution and let S_A and S_B be, respectively, the solutions to the above problem for the following two special loading cases.

$$\begin{aligned} S_A&:\sigma_{\mathrm{I}}=\sigma_{\mathrm{II}}=\sigma_0, \\ S_B&:\sigma_{\mathrm{I}}=-\sigma_{\mathrm{II}}=\sigma_0, \end{aligned} \tag{2.10}$$

† It is important to note that we cannot know *a priori* whether or not the membrane and half-space load each other along this circle. Indeed this question must be answered by the solution. For a discussion on this point see [6] where an analogous situation arises in connection with load transfer from a one-dimensional tension bar to a two-dimensional elastic sheet.

where σ_0 is a constant. It may be observed that S_A and S_B correspond, respectively, to a plane isotropic state of stress (i.e. no θ-variation) and to a plane state of pure shear at infinity. One can easily verify that solution S satisfying the loading condition in (2.9) is

$$S = \frac{(\sigma_{\mathrm{I}}+\sigma_{\mathrm{II}})}{2\sigma_0}S_A+\frac{(\sigma_{\mathrm{I}}-\sigma_{\mathrm{II}})}{2\sigma_0}S_B, \tag{2.11}$$

for arbitrary σ_{I} and σ_{II}.

3. AUXILIARY MEMBRANE AND HALF-SPACE SOLUTIONS

The next step is to obtain some auxiliary solutions for the circular membrane and the half-space both loaded on the disc $\hat{R}$ and the ring $\hat{\pi}$. These loads, which are transmitted from one body to the other, are unknown but satisfy (2.6), (2.7). The displacement conditions in (2.8) will eventually be used to generate integral equations for these unknown transfer loads.

Let the loads transferred to the membrane be defined by (see Fig. 2)

$$\begin{aligned} \hat{\sigma}_{rr} &= p_r/h, \qquad \hat{\sigma}_{r\theta} = p_\theta/h \quad \text{on } \hat{\pi}, \\ \hat{f}_r &= q_r/h, \qquad \hat{f}_\theta = q_\theta/h \quad \text{on } \pi. \end{aligned} \tag{3.1}$$

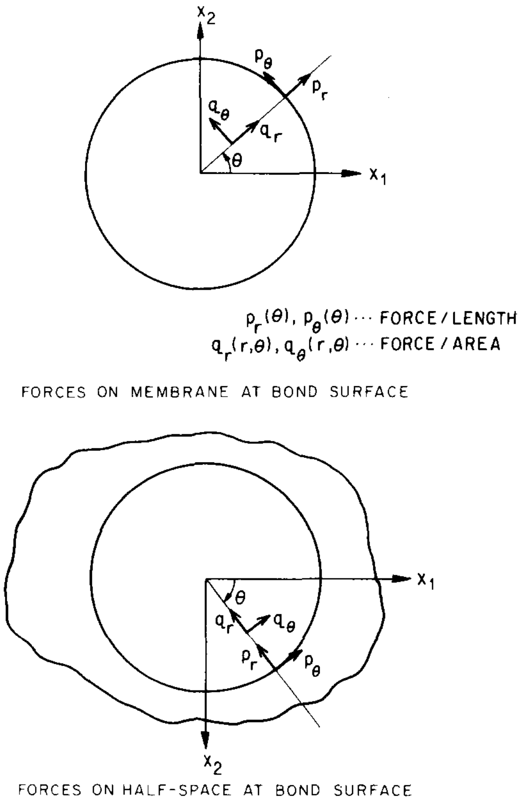

FIG. 2. Bond forces on membrane and half-space.

Then from (3.1) with (2.6) and (2.7) the loads acting on the half-space are

$$\left.\begin{aligned}\sigma_{zr} = p_r\delta(r-a)+q_r, \qquad \sigma_{z\theta} = p_\theta\delta(r-a)+q_\theta,\\ \sigma_{zz} = 0,\end{aligned}\right\} \quad \text{on } \pi. \tag{3.2}$$

Because of the symmetry of the geometry and the functional form of the applied loads the functions p_r, p_θ, q_r and q_θ in (3.2) have a known dependence on θ for cases A and B; for case A:

$$\begin{aligned} p_r(\theta) &= p, & p_\theta(\theta) &= 0,\\ q_r(r,\theta) &= q(r), & q_\theta(r,\theta) &= 0, \end{aligned} \tag{3.3}$$

and for case B:

$$\begin{aligned} p_r(\theta) &= p_1\cos 2\theta, & p_\theta(\theta) &= p_2\sin 2\theta,\\ q_r(r,\theta) &= q_1(r)\cos 2\theta, & q_\theta(r,\theta) &= q_2(r)\sin 2\theta. \end{aligned} \tag{3.4}$$

Membrane solutions

We now list the solutions for the membrane under the loads given by (3.1) with the special forms in equations (3.3) and (3.4) which are appropriate to S_A and S_B.

Case A:

$$\begin{aligned} 2\hat{\mu}\hat{u}_r(r) &= \int_0^a L(r,t)q(t)\,\mathrm{d}t + L(r,a)p,\\ \hat{\sigma}_{rr}(r) &= \int_0^a M(r,t)q(t)\,\mathrm{d}t + M(r,a)p,\\ \hat{\sigma}_{\theta\theta}(r) &= \int_0^a N(r,t)q(t)\,\mathrm{d}t + N(r,a)p,\\ \hat{u}_\theta &= \hat{\sigma}_{r\theta} = 0, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} L(r,t) &= H(t-r)\frac{(1-\hat{\sigma})}{2h}r + H(r-t)\frac{(1-\hat{\sigma})}{2h}\frac{t^2}{r} + \frac{(1-\hat{\sigma})^2}{2h(1+\hat{\sigma})}\left(\frac{t}{a}\right)^2 r,\\ M(r,t) &= H(t-r)\frac{(1+\hat{\sigma})}{2h} - H(r-t)\frac{(1-\hat{\sigma})}{2h}\left(\frac{t}{r}\right)^2 + \frac{(1-\hat{\sigma})}{2h}\left(\frac{t}{a}\right)^2,\\ N(r,t) &= H(t-r)\frac{(1+\hat{\sigma})}{2h} + H(r-t)\frac{(1-\hat{\sigma})}{2h}\left(\frac{t}{r}\right)^2 + \frac{(1-\hat{\sigma})}{2h}\left(\frac{t}{a}\right)^2. \end{aligned} \tag{3.6}$$

The functions L, M and N in (3.6) give $2\hat{\mu}\hat{u}_r(r;t)$, $\hat{\sigma}_{rr}(r;t)$ and $\hat{\sigma}_{\theta\theta}(r;t)$ for a membrane with a concentrated ring body force of magnitude $1/h$, acting in the positive radial direction on the ring $r = t$ and with vanishing tractions at $\hat{\pi}(r = a)$. This solution satisfies the condition of continuous displacement at $r = t$, provides a unit jump discontinuity† in $h\hat{\sigma}_{rr}$ at

† A step function at $r = t$ in $\hat{\sigma}_{rr}$ corresponds to a delta function in the body force $\hat{f}_r$.

$r = t$, and satisfies $\hat{\sigma}_{rr}(a; t) = 0$. The function $H(x)$ is the usual unit step function and is defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases} \tag{3.7}$$

Case B:

$$\begin{aligned}
\frac{2\hat{\mu}\hat{u}_r(r,\theta)}{\cos 2\theta} &= \int_0^a L_{1\alpha}(r,t)q_\alpha(t)\,\mathrm{d}t + L_{1\alpha}(r,a)p_\alpha, \\
\frac{2\hat{\mu}\hat{u}_\theta(r,\theta)}{\sin 2\theta} &= \int_0^a L_{2\alpha}(r,t)q_\alpha(t)\,\mathrm{d}t + L_{2\alpha}(r,a)p_\alpha, \\
\frac{\hat{\sigma}_{rr}(r,\theta)}{\cos 2\theta} &= \int_0^a M_\alpha(r,t)q_\alpha(t)\,\mathrm{d}t + M_\alpha(r,a)p_\alpha, \\
\frac{\hat{\sigma}_{\theta\theta}(r,\theta)}{\cos 2\theta} &= \int_0^a N_\alpha(r,t)q_\alpha(t)\,\mathrm{d}t + N_\alpha(r,a)p_\alpha, \\
\frac{\hat{\sigma}_{r\theta}(r,\theta)}{\sin 2\theta} &= \int_0^a O_\alpha(r,t)q_\alpha(t)\,\mathrm{d}t + O_\alpha(r,a)p_\alpha,
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
hL_{11}(r,t) &= H(t-r)\left[\frac{r}{2} - \frac{\hat{\sigma}}{6}\frac{r^3}{t^2}\right] + H(r-t)\left[-\frac{\hat{\sigma}}{6}\frac{t^4}{r^3} + \frac{1}{2}\frac{t^2}{r}\right] \\
&\quad + r\left[-\frac{\hat{\sigma}}{2}\left(\frac{t}{a}\right)^4 + \frac{(1+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^2\right] + \frac{2\hat{\sigma}}{3(1+\hat{\sigma})}\frac{r^3}{a^2}\left[\hat{\sigma}\left(\frac{t}{a}\right)^4 - \frac{3(1+\hat{\sigma})}{4}\left(\frac{t}{a}\right)^2\right], \\
hL_{12}(r,t) &= H(t-r)\left[-\frac{(1-\hat{\sigma})}{4}r - \frac{\hat{\sigma}}{6}\frac{r^3}{t^2}\right] + H(r-t)\left[\frac{(3+\hat{\sigma})}{12}\frac{t^4}{r^3} - \frac{1}{2}\frac{t^2}{r}\right] \\
&\quad + r\left[\frac{(3+\hat{\sigma})}{4}\left(\frac{t}{a}\right)^4 - \frac{(1+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^2\right] + \frac{\hat{\sigma}}{3(1+\hat{\sigma})}\frac{r^3}{a^2}\left[-(3+\hat{\sigma})\left(\frac{t}{a}\right)^4 + \frac{3(1+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^2\right], \\
hL_{21}(r,t) &= H(t-r)\left[-\frac{r}{2} + \frac{(3+\hat{\sigma})}{12}\frac{r^3}{t^2}\right] + H(r-t)\left[-\frac{\hat{\sigma}}{6}\frac{t^4}{r^3} - \frac{(1-\hat{\sigma})}{4}\frac{t^2}{r}\right] \\
&\quad + r\left[\frac{\hat{\sigma}}{2}\left(\frac{t}{a}\right)^4 - \frac{(1+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^2\right] + \frac{(3+\hat{\sigma})}{3(1+\hat{\sigma})}\frac{r^3}{a^2}\left[-\hat{\sigma}\left(\frac{t}{a}\right)^4 + \frac{3(1+\hat{\sigma})}{4}\left(\frac{t}{a}\right)^2\right], \\
hL_{22}(r,t) &= H(t-r)\left[\frac{(1-\hat{\sigma})}{4}r + \frac{(3+\hat{\sigma})}{12}\frac{r^3}{t^2}\right] + H(r-t)\left[\frac{(3+\hat{\sigma})}{12}\frac{t^4}{r^3} + \frac{(1-\hat{\sigma})}{4}\frac{t^2}{r}\right] \\
&\quad + r\left[-\frac{(3+\hat{\sigma})}{4}\left(\frac{t}{a}\right)^4 + \frac{(1+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^2\right] + \frac{(3+\hat{\sigma})}{3(1+\hat{\sigma})}\frac{r^3}{a^2}\left[\frac{(3+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^4 - \frac{3(1+\hat{\sigma})}{4}\left(\frac{t}{a}\right)^2\right], \\
hM_1(r,t) &= \frac{H(t-r)}{2} + H(r-t)\left[\frac{\hat{\sigma}}{2}\left(\frac{t}{r}\right)^4 - \frac{(1+\hat{\sigma})}{2}\left(\frac{t}{r}\right)^2\right] + \left[-\frac{\hat{\sigma}}{2}\left(\frac{t}{a}\right)^4 + \frac{(1+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^2\right],
\end{aligned} \tag{3.9}$$

$$hM_2(r,t) = -H(t-r)\frac{(1-\hat{\sigma})}{4} + H(r-t)\left[-\frac{(3+\hat{\sigma})}{4}\left(\frac{t}{r}\right)^4 + \frac{(1+\hat{\sigma})}{2}\left(\frac{t}{r}\right)^2\right]$$
$$+\left[\frac{(3+\hat{\sigma})}{4}\left(\frac{t}{a}\right)^4 - \frac{(1+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^2\right],$$

$$hN_1(r,t) = H(t-r)\left[-\frac{1}{2} + \frac{(1+\hat{\sigma})}{2}\left(\frac{r}{t}\right)^2\right] - H(r-t)\frac{\hat{\sigma}}{2}\left(\frac{t}{r}\right)^4$$
$$+\left[\frac{\hat{\sigma}}{2}\left(\frac{t}{a}\right)^4 - \frac{(1+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^2\right] + \left(\frac{r}{a}\right)^2\left[-2\hat{\sigma}\left(\frac{t}{a}\right)^4 + \frac{3(1+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^2\right],$$

$$hN_2(r,t) = H(t-r)\left[\frac{(1-\hat{\sigma})}{4} + \frac{(1+\hat{\sigma})}{2}\left(\frac{r}{t}\right)^2\right] + H(r-t)\left[\frac{(3+\hat{\sigma})}{4}\left(\frac{t}{r}\right)^4\right]$$
$$+\left[-\frac{(3+\hat{\sigma})}{4}\left(\frac{t}{a}\right)^4 + \frac{(1+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^2\right] + \left(\frac{r}{a}\right)^2\left[(3+\hat{\sigma})\left(\frac{t}{a}\right)^4 - \frac{3(1+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^2\right],$$

$$hO_1(r,t) = H(t-r)\left[-\frac{1}{2} + \frac{(1+\hat{\sigma})}{4}\left(\frac{r}{t}\right)^2\right] + H(r-t)\left[\frac{\hat{\sigma}}{2}\left(\frac{t}{r}\right)^4 - \frac{(1+\hat{\sigma})}{4}\left(\frac{t}{r}\right)^2\right]$$
$$+\left[\frac{\hat{\sigma}}{2}\left(\frac{t}{a}\right)^4 - \frac{(1+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^2\right] + \left(\frac{r}{a}\right)^2\left[-\hat{\sigma}\left(\frac{t}{a}\right)^4 + \frac{3(1+\hat{\sigma})}{4}\left(\frac{t}{a}\right)^2\right],$$

$$hO_2(r,t) = H(t-r)\left[\frac{(1-\hat{\sigma})}{4} + \frac{(1+\hat{\sigma})}{4}\left(\frac{r}{t}\right)^2\right] + H(r-t)\left[-\frac{(3+\hat{\sigma})}{4}\left(\frac{t}{r}\right)^4 + \frac{(1+\hat{\sigma})}{4}\left(\frac{t}{r}\right)^2\right]$$
$$+\left[-\frac{(3+\hat{\sigma})}{4}\left(\frac{t}{a}\right)^4 + \frac{(1+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^2\right] + \left(\frac{r}{a}\right)^2\left[\frac{(3+\hat{\sigma})}{2}\left(\frac{t}{a}\right)^4 - \frac{3(1+\hat{\sigma})}{4}\left(\frac{t}{a}\right)^2\right].$$

The functions $L_{1\alpha}$, $L_{2\alpha}$, M_α, N_α and O_α in (3.8) and (3.9) give $2\hat{\mu}\hat{u}_r(r,\theta;t)/\cos 2\theta$, $2\hat{\mu}\hat{u}_\theta(r,\theta;t)/\sin 2\theta$, $\hat{\sigma}_{rr}(r,\theta;t)/\cos 2\theta$, $\hat{\sigma}_{\theta\theta}(r,\theta;t)/\cos 2\theta$ and $\hat{\sigma}_{r\theta}(r,\theta;t)/\sin 2\theta$ for a membrane with concentrated ring body forces acting in the positive radial and tangential directions at $r = t$ with $\cos 2\theta$ and $\sin 2\theta$ variations in θ and with vanishing tractions at $\hat{\pi}$. This solution satisfies the condition of continuous displacements at $r = t$, provides a unit jump discontinuity modified by the appropriate θ-variation in $h\hat{\sigma}_{rr}$ and $h\hat{\sigma}_{\theta\theta}$ at $r = t$ and satisfies $\hat{\sigma}_{rr}(a,\theta;t) = \hat{\sigma}_{r\theta}(a,\theta;t) = 0$.

Half-space solutions

In order to use existing half-space solutions which satisfy the condition of vanishing stress as $|\mathbf{x}| \to \infty$, it is first necessary to separate appropriate uniform stress fields from the half-space solutions sought here. Thus on R, let

$$\{u_i, \varepsilon_{ij}, \sigma_{ij}\} = \{u_i', \varepsilon_{ij}', \sigma_{ij}'\} + \{u_i'', \varepsilon_{ij}'', \sigma_{ij}''\}, \tag{3.10}$$

where for case A:

$$u_r'' = \frac{\sigma_0(1-\sigma)}{2\mu(1+\sigma)}r, \quad u_\theta'' = 0, \quad u_z'' = -\frac{\sigma_0\sigma}{\mu(1+\sigma)}z,$$
$$\sigma_{rr}'' = \sigma_0, \quad \sigma_{\theta\theta}'' = \sigma_0, \quad \sigma_{r\theta}'' = 0, \quad \sigma_{iz}'' = 0, \tag{3.11}$$

and for case B:

$$u_r'' = \frac{\sigma_0}{2\mu} r\cos 2\theta, \qquad u_\theta'' = -\frac{\sigma_0}{2\mu} r\sin 2\theta, \qquad u_z'' = 0,$$
$$\sigma_{rr}'' = \sigma_0 \cos 2\theta, \qquad \sigma_{\theta\theta}'' = -\sigma_0 \cos 2\theta, \qquad \sigma_{r\theta}'' = -\sigma_0 \sin 2\theta, \qquad \sigma_{iz}'' = 0. \tag{3.12}$$

These fields are half-space solutions which satisfy the loading conditions appropriate to S_A and S_B at infinity and induce no tractions on π.

In view of (2.6)–(2.10), (3.2)–(3.4) and (3.10)–(3.12) the residual half-space problems for $\{u_i', \varepsilon_{ij}', \sigma_{ij}'\}$ are governed by the boundary conditions

case A:

$$\sigma_{zr}' = p\delta(r-a) + q(r), \qquad \sigma_{z\theta}' = \sigma_{zz}' = 0 \text{ on } \pi \tag{3.13}$$

case B:

$$\left.\begin{aligned} \sigma_{zr}' &= [p_1\delta(r-a) + q_1(r)]\cos 2\theta \\ \sigma_{z\theta}' &= [p_2\delta(r-a) + q_2(r)]\sin 2\theta, \qquad \sigma_{zz}' = 0 \end{aligned}\right\} \text{ on } \pi \tag{3.14}$$

and for both cases the regularity condition

$$\sigma_{ij}' \to 0 \quad \text{as } |\mathbf{x}| \to \infty. \tag{3.15}$$

The solution to the above traction boundary value problem for the half-space can be obtained from Muki's [7] general solution for arbitrary shear loads on the boundary of a half-space and can be written as

case A:

$$\begin{aligned}
2\mu u_r'(r,z) &= \int_0^a J(r,z,t)q(t)\,\mathrm{d}t + J(r,z,a)p, \\
2\mu u_z'(r,z) &= \int_0^a K(r,z,t)q(t)\,\mathrm{d}t + K(r,z,a)p, \\
\sigma_{rr}'(r,z) &= \int_0^a A(r,z,t)q(t)\,\mathrm{d}t + A(r,z,a)p, \\
\sigma_{\theta\theta}'(r,z) &= \int_0^a B(r,z,t)q(t)\,\mathrm{d}t + B(r,z,a)p, \\
\sigma_{zz}'(r,z) &= \int_0^a C(r,z,t)q(t)\,\mathrm{d}t + C(r,z,a)p, \\
\sigma_{zr}'(r,z) &= \int_0^a D(r,z,t)q(t)\,\mathrm{d}t + D(r,z,a)p, \\
\sigma_{\theta z}' &= \sigma_{r\theta}' = u_\theta' = 0,
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned} J &= (1-\sigma)(I_1^0 - I_{-1}^0) - zI_1^1/2, & K &= -(1-2\sigma)I_0^0 - zI_0^1, \\ A &= 2I_0^1 - zI_0^2 - J/r, & B &= 2\sigma I_0^1 + J/r, \\ C &= zI_0^2, & D &= [-I_1^1 + I_{-1}^1 + z(I_1^2 - I_{-1}^2)]/2, \end{aligned} \tag{3.17}$$

in which

$$I_q^p(r, z, t) = -t \int_0^\infty \mathrm{e}^{-\xi z} \xi^p J_q(\xi r) J_1(\xi t)\, \mathrm{d}\xi, \tag{3.18}$$

and $J_q(x)$ is Bessel's function of the first kind of order q;

case B:

$$\begin{aligned} \frac{2\mu u_r'(r,\theta,z)}{\cos 2\theta} &= \int_0^a J_{1\alpha}(r,z,t) q_\alpha(t)\, \mathrm{d}t + J_{1\alpha}(r,z,a) p_\alpha, \\ \frac{2\mu u_\theta'(r,\theta,z)}{\sin 2\theta} &= \int_0^a J_{2\alpha}(r,z,t) q_\alpha(t)\, \mathrm{d}t + J_{2\alpha}(r,z,a) p_\alpha, \\ \frac{2\mu u_z'(r,\theta,z)}{\cos 2\theta} &= \int_0^a K_\alpha(r,z,t) q_\alpha(t)\, \mathrm{d}t + K_\alpha(r,z,a) p_\alpha, \\ \frac{\sigma_{rr}'(r,\theta,z)}{\cos 2\theta} &= \int_0^a A_\alpha(r,z,t) q_\alpha(t)\, \mathrm{d}t + A_\alpha(r,z,a) p_\alpha, \\ \frac{\sigma_{\theta\theta}'(r,\theta,z)}{\cos 2\theta} &= \int_0^a B_\alpha(r,z,t) q_\alpha(t)\, \mathrm{d}t + B_\alpha(r,z,a) p_\alpha, \\ \frac{\sigma_{zz}'(r,\theta,z)}{\cos 2\theta} &= \int_0^a C_\alpha(r,z,t) q_\alpha(t)\, \mathrm{d}t + C_\alpha(r,z,a) p_\alpha, \\ \frac{\sigma_{\theta z}'(r,\theta,z)}{\sin 2\theta} &= \int_0^a D_\alpha(r,z,t) q_\alpha(t)\, \mathrm{d}t + D_\alpha(r,z,a) p_\alpha, \\ \frac{\sigma_{zr}'(r,\theta,z)}{\cos 2\theta} &= \int_0^a E_\alpha(r,z,t) q_\alpha(t)\, \mathrm{d}t + E_\alpha(r,z,a) p_\alpha, \\ \frac{\sigma_{r\theta}'(r,\theta,z)}{\sin 2\theta} &= \int_0^a F_\alpha(r,z,t) q_\alpha(t)\, \mathrm{d}t + F_\alpha(r,z,a) p_\alpha, \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} J_{\alpha\beta} &= \frac{(2-\sigma)}{2}[K_3^0 + (-1)^{\alpha+\beta} I_1^0] - \frac{\sigma}{2}[(-1)^\beta I_3^0 + (-1)^\alpha K_1^0] \\ &\quad - \frac{z}{4}[(-1)^\beta I_3^1 + (-1)^{\alpha+\beta+1} I_1^1 + K_3^1 + (-1)^\alpha K_1^1], \\ K_\alpha &= -\frac{(1-2\sigma)}{2}[(-1)^\alpha I_2^0 + K_2^0] - \frac{z}{2}[(-1)^\alpha I_2^1 + K_2^1], \end{aligned}$$

$$
\begin{aligned}
A_\alpha &= (-1)^\alpha I_2^1 + K_2^1 - \frac{z}{2}[(-1)^\alpha I_2^2 + K_2^2] - \frac{1}{2r}\{(2-\sigma)[3K_3^0 + (-1)^\alpha I_1^0] \\
&\quad - \sigma[(-1)^\alpha 3I_3^0 + K_1^0] - \frac{z}{2}[(-1)^\alpha 3I_3^1 + (-1)^{\alpha+1} I_1^1 + 3K_3^1 + K_1^1]\}, \\
B_\alpha &= \sigma[(-1)^\alpha I_2^1 + K_2^1] + \frac{1}{2r}\{(2-\sigma)[3K_3^0 + (-1)^\alpha I_1^0] \\
&\quad - \sigma[(-1)^\alpha 3I_3^0 + K_1^0] - \frac{z}{2}[(-1)^\alpha I_3^1 + (-1)^{\alpha+1} I_1^1 + 3K_3^1 + K_1^1]\}, \\
C_\alpha &= \frac{z}{2}[(-1)^\alpha I_2^2 + K_2^2], \\
D_\alpha &= -\tfrac{1}{2}[K_3^1 + (-1)^\alpha I_1^1] + \frac{z}{4}[(-1)^\alpha I_3^2 + K_3^2 + (-1)^\alpha I_1^2 + K_1^2], \\
E_\alpha &= -\tfrac{1}{2}[K_3^1 + (-1)^{\alpha+1} I_1^1] + \frac{z}{4}[(-1)^\alpha I_3^2 + K_3^2 + (-1)^{\alpha+1} I_1^2 - K_1^2], \\
F_\alpha &= \tfrac{1}{2}[(-1)^{\alpha+1} I_2^1 + K_2^1] - \frac{1}{2r}\{(2-\sigma)[3K_3^0 + (-1)^{\alpha+1} I_1^0] \\
&\quad - \sigma[(-1)^\alpha 3I_3^0 - K_1^0] - \frac{z}{2}[(-1)^\alpha 3I_3^1 + (-1)^\alpha I_1^1 + 3K_3^1 - K_1^1]\},
\end{aligned}
\tag{3.20}
$$

in which

$$
K_q^p(r, z, t) = -t \int_0^\infty e^{-\xi z} \xi^p J_3(\xi t) J_q(\xi r)\,d\xi. \tag{3.21}
$$

Clearly from the form of (3.16) and (3.19) the functions J, K, A, B, C and D in (3.17) and $J_{\alpha\beta}$, K_α, A_α, B_α, C_α, D_α, E_α and F_α in (3.20) can be interpreted in terms of solutions for concentrated ring loads applied to the surface of the half-space at $r = t$. This interpretation is similar to that made for the membrane solutions (3.6) and (3.9).

4. INTEGRAL EQUATIONS FOR LOAD TRANSFER FUNCTIONS

The membrane and half-space solutions given in (3.5)–(3.9) and (3.16)–(3.21) in terms of the unknown loads, which arise on each member due to the mutual transfer of bond forces, satisfy for the original problem the conditions (2.6), (2.7) and (2.9). The next step is to derive integral equations governing the unknown load transfer functions by requiring that the above auxiliary solutions satisfy the only remaining condition, namely continuity of displacement in (2.8). We obtain in this manner from the displacement formulas in (2.8), (3.5), (3.6), (3.8)–(3.12) and (3.16)–(3.21) the following equations:

case A:

$$
\begin{aligned}
&\lim_{z\to 0}\left\{\frac{1}{2\mu}\int_0^a J(r, z, t)q(t)\,dt + \frac{J(r, z, a)}{2\mu}p\right\} + \frac{\sigma_0(1-\sigma)}{2\mu(1+\sigma)}r \\
&\quad = \frac{1}{2\hat{\mu}}\int_0^a L(r, t)q(t)\,dt + \frac{L(r, a)}{2\hat{\mu}}p \quad \text{for } 0 \le r < a,
\end{aligned}
\tag{4.1}
$$

case B:

$$\lim_{z\to 0}\left\{\frac{1}{2\mu}\int_0^a J_{\alpha\beta}(r,z,t)q_\beta(t)\,\mathrm{d}t+\frac{1}{2\mu}J_{\alpha\beta}(r,z,a)p_\beta\right\}$$
$$+(-1)^{\alpha+1}\frac{\sigma_0 r}{2\mu}=\frac{1}{2\hat{\mu}}\int_0^a L_{\alpha\beta}(r,t)q_\beta(t)\,\mathrm{d}t+\frac{L_{\alpha\beta}(r,a)p_\beta}{2\hat{\mu}}\quad\text{for } 0\le r<a. \tag{4.2}$$

It would be desirable at this point to interchange the limiting process $z \to 0$ with that inherent in the integrals in (4.1) and (4.2). Normally this could be done formally and the solution could be obtained subject to *a posteriori* verification. However, equations (4.1) and (4.2) are somewhat unusual in that the quantities p and p_γ as well as q and q_γ are, at present, unknown. That is, the question of whether or not the membrane and the half-space transmit forces to each other along the ring $r = a$ must be settled from these equations. But this question is related to the singular nature of the unknown functions q and q_γ and the kernel functions in (4.1) and (4.2). Therefore it is necessary to make certain assumptions about the functions q and q_γ and, then on the basis of these assumptions, to show that the interchange of limits is permissible. Thus we assume the functions $q(r)$ and $q_\gamma(r)$ have the form

$$q(r)=\frac{\tilde{q}(r)}{r^\alpha(a-r)^\beta},\qquad q_\gamma(r)=\frac{\tilde{q}_\gamma(r)}{r^{\alpha_\gamma}(a-r)^{\beta_\gamma}},\quad\text{(no sum)}$$
$$\gamma=1,2,\qquad 0\le\alpha,\beta,\alpha_\gamma,\beta_\gamma<1, \tag{4.3}$$

where $\tilde{q}(r)$ and $\tilde{q}_\gamma(r)$ are uniformly Holder continuous on $[0, a]$. This assumption implies that

$$q(r)=O(r^{-\alpha})\quad\text{as } r\to 0,\qquad O[(a-r)^{-\beta}]\quad\text{as } r\to a,$$
$$q_\gamma(r)=O(r^{-\alpha_\gamma})\quad\text{as } r\to 0,\qquad O[(a-r)^{-\beta_\gamma}]\quad\text{as } r\to a, \tag{4.4}$$

and insures that q and q_γ are integrable on $[0, a]$.

If use is now made of the identity (see Ref. [8])

$$\int_0^\infty \mathrm{e}^{-\xi z}J_\gamma(\xi t)J_\gamma(\xi r)\,\mathrm{d}\xi=\frac{1}{\pi(rt)^{\frac{1}{2}}}Q_{\gamma-\frac{1}{2}}\left(\frac{z^2+r^2+t^2}{2rt}\right), \tag{4.5}$$

where $Q_\delta(x)$ is Legendre's function of the second kind, as well as the identity (see Ref. [9])

$$\int_0^\infty J_1(\xi a)J_3(\xi b)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi=-4\frac{a}{b}\int_0^\infty J_2(\xi a)J_2(\xi b)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi$$
$$+3\int_0^\infty J_1(\xi a)J_1(\xi b)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi-\frac{4z}{b}\int_0^\infty J_1(\xi a)J_2(\xi b)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi, \tag{4.6}$$

then, from (4.3), (3.17), (3.18), (3.20) and (3.21) one can justify taking the limit $z \to 0$ inside the integrals, and the integral equations (4.1) and (4.2) become

case A:

$$\int_0^a\left\{J(r,0,t)-\frac{\mu}{\hat{\mu}}L(r,t)\right\}q(t)\,\mathrm{d}t+\left\{J(r,0,a)-\frac{\mu}{\hat{\mu}}L(r,a)\right\}p$$
$$=-\frac{\sigma_0(1-\sigma)}{(1+\sigma)}r,\qquad 0\le r<a, \tag{4.7}$$

where

$$J(r, 0, t) = -(1-\sigma)\frac{2}{\pi}\frac{t^{\frac{1}{2}}}{r^{\frac{1}{2}}}Q_{\frac{1}{2}}\left(\frac{r^2+t^2}{2rt}\right), \tag{4.8}$$

case B:

$$\int_0^a \left\{J_{\alpha\beta}(r, 0, t)-\frac{\mu}{\bar{\mu}}L_{\alpha\beta}(r, t\right\} q_\beta(t)\,dt + \left\{J_{\alpha\beta}(r, 0, a)-\frac{\mu}{\bar{\mu}}L_{\alpha\beta}(r, a)\right\} p_\beta$$

$$= (-1)^\alpha \sigma_0 r, \qquad 0 \le r < a, \tag{4.9}$$

where

$$\begin{aligned} J_{\alpha\beta}(r, 0, t) = \frac{(2-\sigma)}{2\pi}&\left[-\frac{t^{\frac{1}{2}}}{r^{\frac{1}{2}}}Q_{\frac{3}{2}}\left(\frac{r^2+t^2}{2rt}\right)+(-1)^{\alpha+\beta+1}\frac{t^{\frac{1}{2}}}{r^{\frac{1}{2}}}Q_{\frac{1}{2}}\left(\frac{r^2+t^2}{2rt}\right)\right] \\ -\frac{\sigma}{2\pi}&\left[(-1)^\beta\frac{4t^{\frac{3}{2}}}{r^{\frac{1}{2}}}Q_{\frac{3}{2}}\left(\frac{r^2+t^2}{2rt}\right)+(-1)^{\beta+1}\frac{3t^{\frac{1}{2}}}{r^{\frac{1}{2}}}Q_{\frac{1}{2}}\left(\frac{r^2+t^2}{2rt}\right)\right. \\ &\left.+(-1)^\alpha\frac{4r^{\frac{1}{2}}}{t^{\frac{1}{2}}}Q_{\frac{3}{2}}\left(\frac{r^2+t^2}{2rt}\right)+(-1)^{\alpha+1}\frac{3t^{\frac{1}{2}}}{r^{\frac{1}{2}}}Q_{\frac{1}{2}}\left(\frac{r^2+t^2}{2rt}\right)\right]. \end{aligned} \tag{4.10}$$

5. INVESTIGATION OF KERNELS AND DETERMINATION OF UNKNOWN EDGE LOADS

In order to determine the unknown edge transfer functions p and p_α appearing in the integral equations (4.7) and (4.9), one must first study the singular nature of their kernels given by (4.8), (4.10), (3.6) and (3.9). Clearly from (3.6) and (3.9) the parts of the kernels given by L and $L_{\alpha\beta}$ contribute only finite discontinuities. If x is defined by

$$x = \frac{r^2+t^2}{2rt}, \tag{5.1}$$

then the singularities in the remaining parts of the kernels can be found by examining their behavior as $x \to \infty$ (corresponding to $r \to 0$ or $t \to 0$) and as $x \to 1$ (corresponding to $t \to r$). From the results in [10, pp. 37, 153, 196] it follows that

$$\begin{aligned} Q_\nu(x) &= O(x^{-(1+\nu)} \quad \text{as } x \to \infty, \\ Q_\nu(x) &= -\frac{1}{2}\ln\left(\frac{x-1}{2}\right)-\gamma-\psi(\nu+1)+o(1) \quad \text{as } x \to 1, \end{aligned} \tag{5.2}$$

where γ is Euler's constant, and ψ is the psi function, which satisfies the identity

$$\psi(n+\tfrac{1}{2}) = -\gamma-2\ln 2+2\sum_{l=0}^{n-1}\frac{1}{2l+1}, \qquad n = 1, 2, \ldots. \tag{5.3}$$

Therefore from (4.8) and (4.10),

$$
\begin{aligned}
&J(r,0,t),\, J_{\alpha\beta}(r,0,t) \to 0 \quad \text{as } r \to 0 \quad \text{or } t \to 0,\\
&J(r,0,t) = \frac{2(1-\sigma)}{\pi}\left[\ln\frac{|t-r|}{2r} - 2\ln 2 + 2\right] + o(1) \quad \text{as } t \to r,\\
&J_{\alpha\beta}(r,0,t) = \left\{\ln\frac{|t-r|}{2r} - 2\ln 2\right\}\{2-\sigma[1+(-1)^{\alpha+1}]\}\frac{\delta_{\alpha\beta}}{\pi}\\
&\quad + \frac{(2-\sigma)}{\pi}\left[\frac{23}{15} + (-1)^{\alpha+\beta}\right] - \frac{14\sigma}{3\pi}\delta_{\alpha\beta}(-1)^{\alpha+1} + o(1) \quad \text{as } t \to r, \quad \text{no sum.}
\end{aligned}
\tag{5.4}
$$

Let $G(r, t)$ and $G_{\alpha\beta}(r, t)$ be defined through

$$
\begin{aligned}
G(r,t) &= J(r,0,t) - \frac{2(1-\sigma)}{\pi}\ln\frac{|t-r|}{t+r},\\
G_{\alpha\beta}(r,t) &= J_{\alpha\beta}(r,0,t) - \ln\frac{|t-r|}{t+r}\{2-\sigma[1+(-1)^{\alpha+1}]\}\frac{\delta_{\alpha\beta}}{\pi}, \quad \text{no sum.}
\end{aligned}
\tag{5.5}
$$

Then from (5.4) and (5.5) $G(r, t)$ and $G_{\alpha\beta}(r, t)$ are bounded and continuous on $0 < r \le a$, $0 < t \le a$ and for any fixed direction of approach they have a well defined limit as $(r, t) \to (0, 0)$. In particular

$$
G(r,r) = \frac{2(1-\sigma)}{\pi}(-2\ln 2 + 2),\; G(0,t) = 0,
$$

$$
\begin{aligned}
G_{\alpha\beta}(r,r) &= \frac{-2\ln 2}{\pi}\{2-\sigma[1+(-1)^{\alpha+1}]\}\delta_{\alpha\beta} + \frac{(2-\sigma)}{\pi}\left[\frac{23}{15} + (-1)^{\alpha+\beta}\right]\\
&\quad - \frac{14\sigma}{3\pi}\delta_{\alpha\beta}(-1)^{\alpha+1},\; G_{\alpha\beta}(0,t) = 0, \quad \text{no sum.}
\end{aligned}
\tag{5.6}
$$

Using (5.5) in (4.7)–(4.10), recalling the properties of G, $G_{\alpha\beta}$, L and $L_{\alpha\beta}$ and of q and q_α given in (4.3) and (4.4), we obtain for

case A:

$$
\frac{2(1-\sigma)}{\pi}\ln\frac{|a-r|}{a+r}p + O(1) = 0 \quad \text{as } r \to a, \tag{5.7}
$$

case B:

$$
\ln\frac{|a-r|}{a+r}\{2-\sigma[1+(-1)^{\alpha+1}]\}\frac{\delta_{\alpha\beta}}{\pi}p_\beta + O(1) = 0 \quad \text{as } r \to a, \quad \text{no sum on } \alpha. \tag{5.8}
$$

These equations imply that

$$
p,\, p_\beta = 0, \tag{5.9}
$$

(since σ is restricted to the range $-1 < \sigma < \frac{1}{2}$) and, hence no concentrated line-load transfer between the membrane and the half-space occurs at the ring $r = a$. The resulting integral equations now appear as

case A:

$$\int_0^a \left[2\frac{(1-\sigma)}{\pi}\ln\frac{|t-r|}{t+r} + G(r,t) - \frac{\mu}{\hat{\mu}}L(r,t)\right] q(t)\,\mathrm{d}t = -\frac{\sigma_0(1-\sigma)}{(1+\sigma)}r, \qquad 0 \le r \le a, \tag{5.10}$$

case B:

$$\begin{aligned}\int_0^a \left[-\frac{2(1-\sigma)}{\pi}\ln\frac{|t-r|}{t+r}\delta_{1\beta} - G_{1\beta}(r,t) + \frac{\mu}{\hat{\mu}}L_{1\beta}(r,t)\right] q_\beta(t)\,\mathrm{d}t &= \sigma_0 r,\\ \int_0^a \left[-\frac{2}{\pi}\ln\frac{|t-r|}{t+r}\delta_{2\beta} - G_{2\beta}(r,t) + \frac{\mu}{\hat{\mu}}L_{2\beta}(r,t)\right] q_\beta(t)\,\mathrm{d}t &= -\sigma_0 r, \qquad 0 \le r \le a.\end{aligned} \tag{5.11}$$

These are Fredholm integral equations of the first kind with logarithmically singular kernels.

6. EXACT SOLUTION—INEXTENSIBLE MEMBRANE

The integral equations (5.10) and (5.11) can be solved exactly, and the complete solution in the half-space can be obtained for the limiting case of an inextensible membrane, i.e. when $\hat{\mu} \to \infty$. In this limit the kernels are considerably simplified. At this point it is convenient to return to equations (4.7)–(4.10) with p and p_β zero. Hence the integral equations become

case A:

$$\int_0^a J(r,0,t)q(t)\,\mathrm{d}t = -\frac{\sigma_0(1-\sigma)}{(1+\sigma)}r, \tag{6.1}$$

case B:

$$\int_0^a J_{\alpha\beta}(r,0,t)q_\beta(t)\,\mathrm{d}t = (-1)^\alpha \sigma_0 r. \tag{6.2}$$

Next assume (subject to later verification) for the inextensible membrane that q_1 and q_2 in case B are not independent, but satisfy

$$q_2(t) = -q_1(t). \tag{6.3}$$

Then the pair of integral equations in (6.2) reduce to a single equation, which is of the same form as integral equation (6.1) for case A. Hence for both case A and B the relevant integral equation is

$$\int_0^a J(r,0,t)f(t)\,\mathrm{d}t = cr, \tag{6.4}$$

where for

case A:

$$f(t) = q(t), \qquad c = -\frac{\sigma_0(1-\sigma)}{(1+\sigma)},$$

case B:

$$f(t) = q_1(t) = -q_2(t), \qquad c = -\frac{2\sigma_0(1-\sigma)}{(2-\sigma)}. \tag{6.5}$$

By (4.5) and (4.8), equation (6.4) can be written as

$$\int_0^a tf(t)\left\{\int_0^\infty J_1(\xi t)J_1(\xi r)\,\mathrm{d}\xi\right\}\mathrm{d}t = -\frac{cr}{2(1-\sigma)}. \tag{6.6}$$

This integral equation can be inverted by use of equations (3.1), (3.2), (5.1) and (5.13), given in Popov [11]. The use of these results with (6.5) and (6.6) leads to

case A:

$$q(r) = \frac{2\sigma_0 r}{\pi(1+\sigma)(a^2-r^2)^{\frac{1}{2}}},$$

case B: (6.7)

$$q_1(r) = -q_2(r) = \frac{4\sigma_0 r}{\pi(2-\sigma)(a^2-r^2)^{\frac{1}{2}}}.$$

This result for case A is in agreement† with equation (39) of [3], which considers by an entirely different method an inextensible elliptical membrane bonded to an elastic half-space. The results in [3] can also be extended to produce agreement with those in (6.7) for case B.

The displacement and stress fields in the half-space now follow from the use of (6.7) in (3.16)–(3.21), and after an interchange of the order of integrals followed by the use of [18, p. 688]

$$\begin{aligned}\int_0^a \frac{t^2 J_1(\xi t)}{(a^2-t^2)^{\frac{1}{2}}}\,\mathrm{d}t &= \frac{a^{\frac{3}{2}}\Gamma(\frac{1}{2})}{\sqrt{2}}\xi^{-\frac{1}{2}}J_{\frac{3}{2}}(\xi a),\\ J_{\frac{3}{2}}(\xi a) &= \sqrt{\left(\frac{2}{\pi\xi a}\right)}\left[\frac{\sin(\xi a)}{\xi a}-\cos(\xi a)\right],\end{aligned} \tag{6.8}$$

they appear as

case A:

$$\begin{aligned}
u_r'(r,z) &= \frac{\sigma_0 a}{\mu(1+\sigma)\pi}\left[-2(1-\sigma)H_0^1+\frac{z}{2a}H_1^1\right],\\
u_z'(r,z) &= \frac{\sigma_0 a}{\mu(1+\sigma)\pi}\left[(1-2\sigma)H_0^0+\frac{z}{a}H_1^0\right],\\
\sigma_{rr}'(r,z) &= \frac{2\sigma_0 a}{\pi(1+\sigma)}\left[-\frac{2}{a}H_1^0+\frac{2(1-\sigma)}{r}H_0^1+\frac{z}{a^2}H_2^0-\frac{z}{2ra}H_1^1\right],\\
\sigma_{zz}'(r,z) &= -\frac{2\sigma_0 z}{\pi(1+\sigma)a}H_2^0,\\
\sigma_{rz}'(r,z) &= \frac{2\sigma_0}{\pi(1+\sigma)}\left[H_1^1-\frac{z}{a}H_2^1\right],\\
\sigma_{\theta\theta}'(r,z) &= \frac{2\sigma_0 a}{\pi(1+\sigma)}\left[-\frac{2\sigma}{a}H_1^0-\frac{2(1-\sigma)}{r}H_0^1+\frac{z}{2ra}H_1^1\right],
\end{aligned} \tag{6.9}$$

† The agreement follows after some obvious algebraic sign errors in equation (39) of [3] are corrected.

case B:

$$u_r'(r,\theta,z) = -\frac{2\sigma_0 a\cos 2\theta}{\mu\pi(2-\sigma)}\left[\sigma H_0^3+(2-\sigma)H_0^1+\frac{z}{2a}(H_1^3+H_1^1)\right],$$

$$u_\theta'(r,\theta,z) = -\frac{2\sigma_0 a\sin 2\theta}{\mu\pi(2-\sigma)}\left[\sigma H_0^3-(2-\sigma)H_0^1+\frac{z}{2a}(H_1^3-H_1^1)\right],$$

$$u_z'(r,\theta,z) = -\frac{2\sigma_0 a\cos 2\theta}{\mu\pi(2-\sigma)}\left[(1-2\sigma)H_0^2+\frac{z}{a}H_1^2\right],$$

$$\sigma_{rr}'(r,\theta,z) = -\frac{4\sigma_0 a\cos 2\theta}{\pi(2-\sigma)}\left[-\frac{2}{a}H_1^2-\frac{3\sigma}{r}H_0^3+\frac{(2-\sigma)}{r}H_0^1+\frac{z}{a^2}H_2^2+\frac{z}{2ra}(-3H_1^3+H_1^1)\right],$$

$$\sigma_{\theta\theta}'(r,\theta,z) = -\frac{4\sigma_0 a\cos 2\theta}{\pi(2-\sigma)}\left[-\frac{2\sigma}{a}H_1^2+\frac{3\sigma}{r}H_0^3-\frac{(2-\sigma)}{r}H_0^1+\frac{3z}{2ra}H_1^3-\frac{z}{2ra}H_1^1\right], \tag{6.10}$$

$$\sigma_{zz}'(r,\theta,z) = \frac{4\sigma_0 z\cos 2\theta}{\pi a(2-\sigma)}H_2^2,$$

$$\sigma_{\theta z}'(r,\theta,z) = -\frac{2\sigma_0 a\sin 2\theta}{\pi(2-\sigma)}\left[\frac{2}{a}H_1^1-\frac{z}{a^2}(H_2^3+H_2^1)\right],$$

$$\sigma_{zr}'(r,\theta,z) = -\frac{2\sigma_0 a\cos 2\theta}{\pi(2-\sigma)}\left[-\frac{2}{a}H_1^1-\frac{z}{a^2}(H_2^3-H_2^1)\right],$$

$$\sigma_{r\theta}'(r,\theta,z) = -\frac{4\sigma_0 a\sin 2\theta}{\pi(2-\sigma)}\left[\frac{1}{a}H_1^2-\frac{3\sigma}{r}H_0^3-\frac{(2-\sigma)}{r}H_0^1-\frac{3z}{2ra}H_1^3-\frac{z}{2ra}H_1^1\right],$$

where

$$H_n^m(r,z) = \int_0^\infty \xi^{n-1}\left(\frac{\sin\xi}{\xi}-\cos\xi\right)J_m\left(\xi\frac{r}{a}\right)\mathrm{e}^{-\xi z/a}\,\mathrm{d}\xi,\ z>0. \tag{6.11}$$

The integrals in (6.9)–(6.11) have been evaluated exactly in terms of elementary functions and are given in the Appendix.

The *a posteriori* verification of the solution given here easily follows provided due care is given to the evaluation of the boundary conditions. Some of the integrals in (6.10) do not converge when $z = 0$, but their asymptotic values as $z \to 0$ are found to be such that the solution takes the proper boundary values as $z \to 0$.

7. NUMERICAL SOLUTION OF THE INTEGRAL EQUATIONS FOR ELASTIC MEMBRANES

When the membrane is elastic, the integral equations (5.10), (5.11) for the load transfer functions must be solved numerically. For brevity we restrict our attention henceforth to (5.10) only, i.e. to case A (isotropic stress at infinity), but the techniques used apply equally well to (5.11).

Several techniques have been developed to overcome the difficulties arising in the numerical treatment of equations of the first kind [13–16]. The method we employ [16] replaces an integral equation of the type

$$\int_0^1 K(x,y)f(y)\,\mathrm{d}y = g(x), \tag{7.1}$$

by one of the type

$$\alpha f(x, \alpha)+\int_0^1 K(x, y) f(y, \alpha)\, \mathrm{d}y = g(x), \tag{7.2}$$

in which α is a parameter. This equation of the second kind can be successfully handled numerically to give $f(x, \alpha)$ for various values of α, and the desired solution is obtained in the limit as $\alpha \to 0$ (see [16]). Although this technique in theory produces the desired result, the value of $\alpha \neq 0$ needed to give a solution of (7.2) sufficiently close to the solution of (7.1) is not known. Hence the practical application of this technique depends on the rapidity of the convergence in α.

Before proceeding to the numerical solution of (5.10) it is convenient to put the equation in non-dimensional form and to define a new unknown function which is not singular. Since the singular part of the kernel in (5.10) is the same as that in (6.6) for the inextensible case, the singular nature of the solution of (5.10) is assumed to be the same as that in (6.7). Thus, let

$$\rho = r/a, \qquad \zeta = t/a, \qquad \eta = \mu/\hat{\mu}, \qquad \gamma = h/a, \tag{7.3}$$

and define $\phi(\rho)$ by

$$q(r) = \frac{\phi(\rho)\sigma_0}{(1-\rho)^{\frac{1}{2}}}, \tag{7.4}$$

in which $\phi(\rho)$ is assumed to be continuous for ρ in $[0, 1]$. It should be noted that $q(r)$ in (7.4) is a special case of the form previously assumed in (4.3).

A final change of variables

$$\begin{gathered} \tau = (1-\zeta)^{\frac{1}{2}}, \qquad \iota = (1-\rho)^{\frac{1}{2}}, \qquad \phi(\rho) = \Phi(\iota) \\ g(\iota, \tau ; \sigma) = G[(1-\iota^2)a, (1-\tau^2)a], \qquad l(\iota, \tau ; \hat{\sigma}) = \gamma L[(1-\iota^2)a, (1-\tau^2)a] \end{gathered} \tag{7.5}$$

removes the power singularity which arises in the integrand of (5.10) and yields in the place of (5.10)

$$\begin{aligned} &\frac{2(1-\sigma)}{\pi}\int_0^1 \ln\left[\frac{|\iota^2-\tau^2|}{2-\iota^2-\tau^2}\right]\Phi(\tau)\,\mathrm{d}\tau+\int_0^1\left[g(\iota, \tau ; \sigma)-\frac{\eta}{\gamma}\, l(\iota, \tau ; \hat{\sigma})\right]\Phi(\tau)\,\mathrm{d}\tau \\ &\quad = -\frac{(1-\sigma)(1-\iota^2)}{2(1+\sigma)} \qquad (0 \leq \iota \leq 1). \end{aligned} \tag{7.6}$$

By use of standard techniques (see [19]) equation (7.6), with the additional term $\alpha\Phi(\iota, \alpha)$ to put it in the form of (7.2), is replaced by a system of linear algebraic equations in which the integrals are replaced by appropriate quadrature sums. The integral with the logarithmic term is approximated by replacing $\Phi(\tau, \alpha)$ by a function that is piecewise linear in $[0, 1]$ (i.e. linear over each partition of $[0, 1]$). Then the integration can be carried out analytically over each partition. The second integral is approximated by a quadrature formula using the trapezoidal rule. The resulting equation is then evaluated at each of the nodes of the uniform range of integration.

The numerical solution for $\Phi(\iota)$ was obtained for various values of the material and geometrical parameters η, γ, defined in (7.3), and the two Poissons ratios $\sigma, \hat{\sigma}$. Since η and γ appear in (7.6) only through their ratio, η/γ, which is a measure of the effective stiffness of the half-space relative to the stiffness of the membrane, the three parameters, η/γ, σ and $\hat{\sigma}$, are sufficient to reveal the effect of the materials and geometry on the load transfer function.

Satisfactory convergence of the numerical computations was obtained by use of fourteen intervals in the quadrature formula for all values of the parameter α. The convergence in α was such that suitable results were achieved with $\alpha = 0{\cdot}0001$. With this number of intervals and value of α, $\Phi(\iota, \alpha)$ for $\eta = 0$ was in agreement to within 1 per cent of the exact results given in Section 6 for the inextensible membrane.

It was found by taking advantage of the result† $\Phi(1) = 0$ that the numerical solution of (7.6) was stable, i.e. the correct numerical solution was obtained from the above technique with $\alpha = 0$. This stability of solution is evidently due to the presence of the mildly singular kernel in (7.6). This result significantly reduced the amount of computation for the elastic membrane solutions, since a sequence of solutions for various values of α was averted.

The function $\phi(\rho)$ defined in (7.4) is shown in Figs. 3–5 for various values of η/γ, σ and $\hat{\sigma}$. Figure 3 shows the dependence of $\phi(\rho)$ on η/γ for fixed σ and $\hat{\sigma}$. The curve for $\eta/\gamma = 0$ agrees with the exact results in Section 6 for an inextensible membrane. Figure 4 shows the dependence on σ for fixed η/γ and $\hat{\sigma}$ and Fig. 5 shows the result when $\hat{\sigma}$ varies for fixed η/γ and σ.

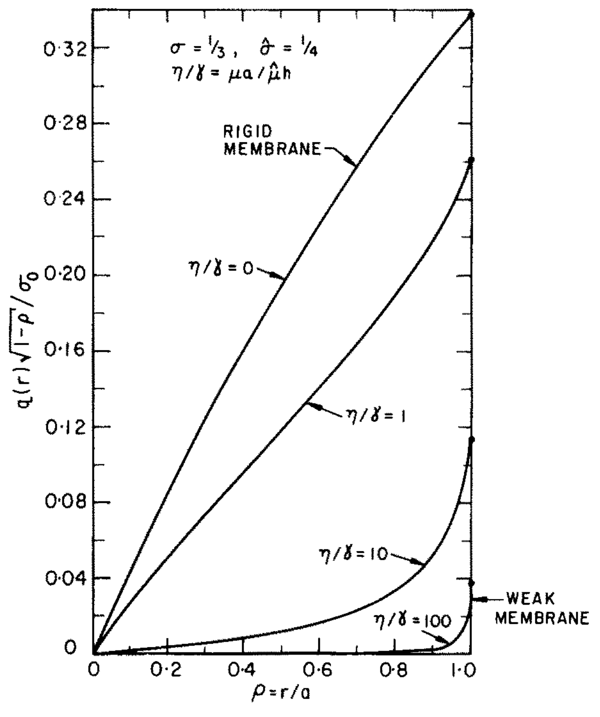

FIG. 3. Dependence of load transfer function on ρ for various η/γ—isotropic loading.

† One can verify from the half-space solution in [7] that when the boundary loading is on the circular region $\hat{R}$ then non-zero shearing tractions at $r = 0$ are identified only with solutions that have a θ-dependence of $\cos\theta$ or $\sin\theta$.

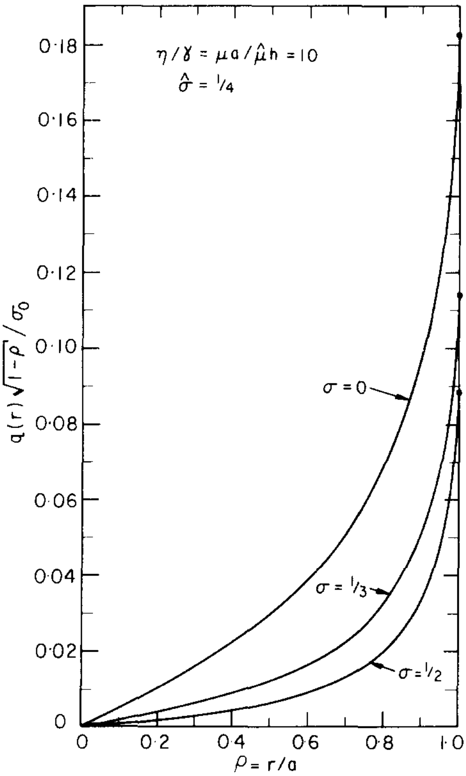

FIG. 4. Dependence of load transfer function on ρ for various σ.

8. NUMERICAL COMPUTATION OF STRESS

In order to illustrate the disturbance of the stress field in the half-space, which results from the attachment of the elastic membrane, we compute $\sigma'_{rr}(r, z)$ in the half-space along $r = 0$ as a function of z and along $z = 0$ as a function of r. Also $\hat{\sigma}_{rr}(r)$ is computed for the membrane. From (3.16) (with $p = 0$), (3.17) and (3.18)

$$\sigma'_{rr}(r, z) = \int_0^a A(r, z, t)q(t)\,\mathrm{d}t. \tag{8.1}$$

By use of asymptotic analysis of $A(r, z, t)$ at its singular points $r, t \to 0$ and $z = 0, t \to r$ in conjunction with the numerical solution for $q(t)$ obtained in the previous section the integral in (8.1) was evaluated numerically. The details of this analysis may be found in [21]. A notable result is that σ'_{rr} at $z = 0$ has a finite limit as $r \to a$ from $r < a$ and has a half-power singularity as $r \to a$ from $r > a$. This same singular behavior can be observed from the exact solution (6.9) for the inextensible membrane.

Figure 6(a) and 6(b) show the half-space radial stress σ'_{rr}/σ_0 for $r = 0$ as a function of z for various choices of the parameters η/γ, σ and $\hat{\sigma}$. These figures indicate that the maximum change from the uniform stress state occurs for $\eta/\gamma \to 0$, $\hat{\sigma} = \frac{1}{2}$ and $\sigma = 0$, i.e. for a relatively stiff, thick, incompressible membrane on a half-space with zero Poissons

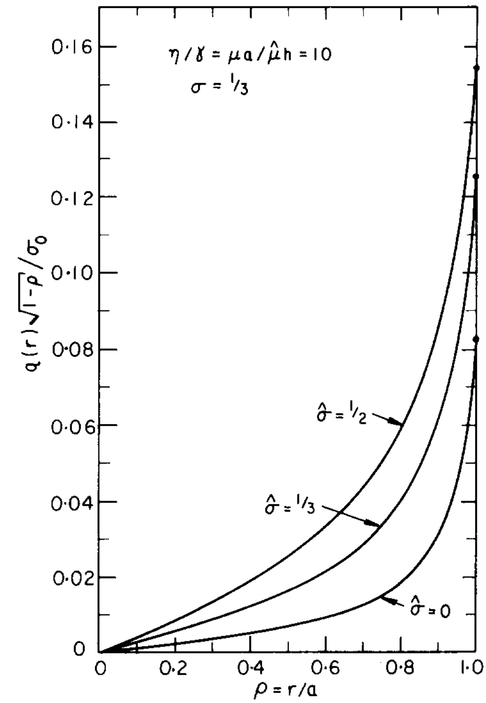

FIG. 5. Dependence of load transfer function on ρ for various $\hat{\sigma}$.

ratio. The disturbance vanishes as $\eta/\gamma \to \infty$. The effect on the radial stress is negligible at depths greater than two or three membrane radii.

Figures 7(a) and 7(b) display the half-space radial stress for $z = 0$ as a function of r for various η/γ, σ and $\hat{\sigma}$. The effect on the radial stress is also negligible at distances more remote than three or four radii from the center of the membrane.

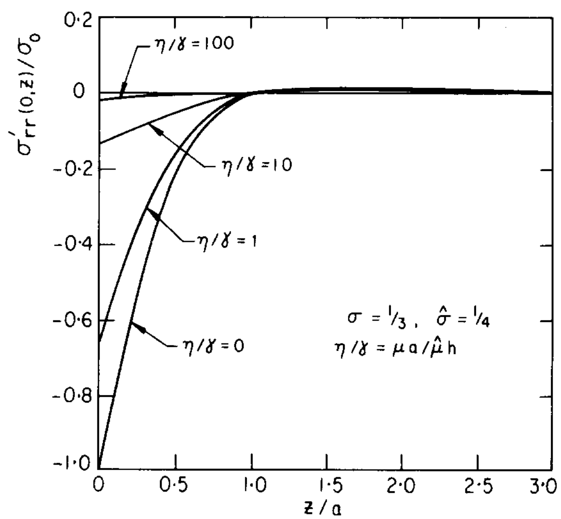

FIG. 6(a). Dependence of half-space σ'_{rr}/σ_0 at $r = 0$ on z/a for various η/γ.

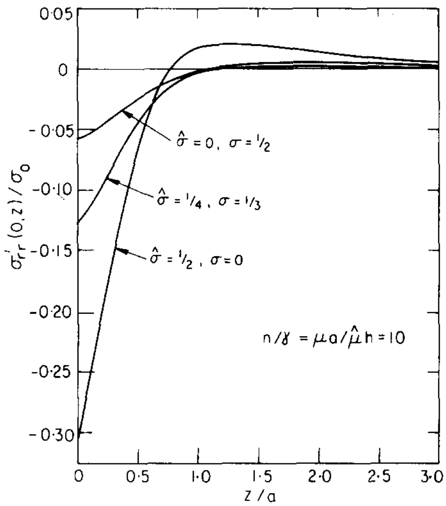

FIG. 6(b). Dependence of half-space σ'_{rr}/σ_0 at $r = 0$ on z/a for various $\hat{\sigma}, \sigma$.

The radial stress $\hat{\sigma}_{rr}$ in the membrane was also evaluated numerically for $0 \le r \le a$ from (3.5) with $p = 0$, (3.6), and the numerical solution for $q(t)$. No special difficulties arose in these calculations after (7.4) and (7.5) were used to remove the singularity in the load transfer function from the integrand.

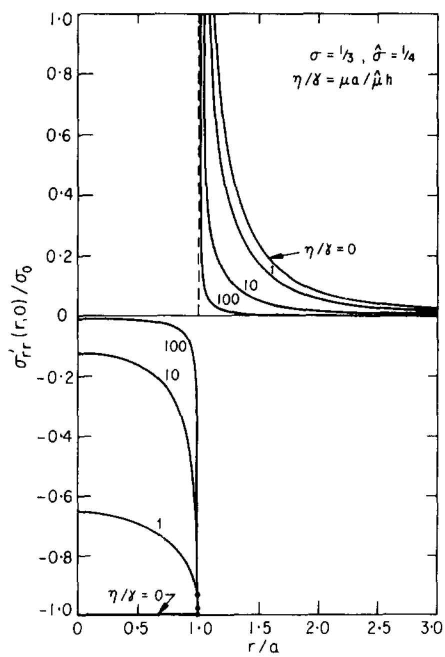

FIG. 7(a). Dependence of half-space stress σ'_{rr}/σ_0 at $z = 0$ on r/a for various η/γ.

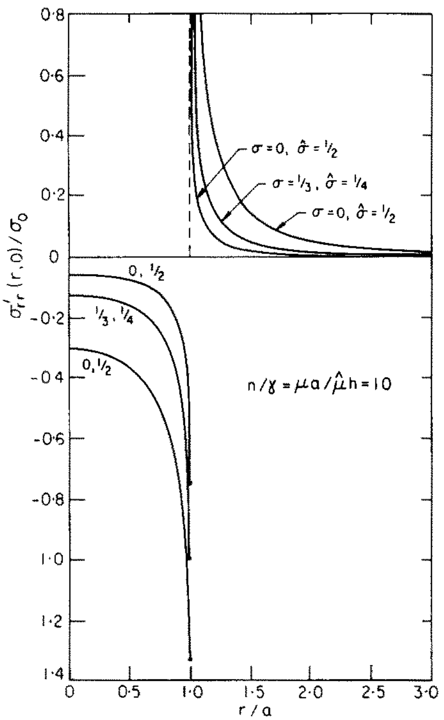

FIG. 7(b). Dependence of half-space stress σ'_{rr}/σ_0 at $z = 0$ on r/a for various $\sigma, \hat{\sigma}$.

Figures 8(a) and 8(b) show the membrane stress $\hat{\sigma}_{rr}$ as a function of r for various η/γ, σ and $\hat{\sigma}$. Note that $\hat{\sigma}_{rr} \to 0$ as $r \to a$ as required by the boundary conditions and the fact that no concentrated load transfer occurs at the edge of the membrane (i.e. $p = 0$). For relatively weak membranes ($\eta/\gamma \to \infty$) the change in $\hat{\sigma}_{rr}$ with r occurs noticeably only near $r = a$.

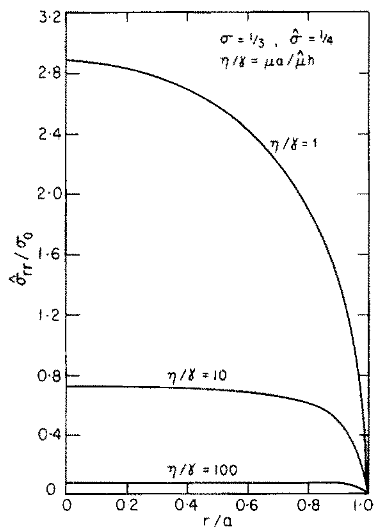

FIG. 8(a). Dependence of membrane stress $\hat{\sigma}_{rr}/\sigma_0$ on r/a for various η/γ.

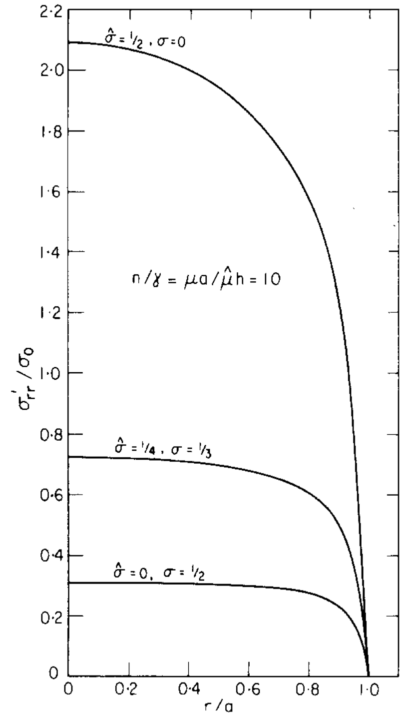

FIG. 8(b). Dependence of membrane stress $\hat{\sigma}_{rr}/\sigma_0$ on r/a for various $\hat{\sigma}, \sigma$.

9. APPLICATION TO STRAIN MEASUREMENTS

As seen in the previous section the attachment of an elastic membrane to an elastic body can significantly perturb the state of deformation in the vicinity of the membrane provided the effective stiffness ratio is small enough. This result has implications with regard to the accuracy of the measurement of strain in a relatively weak elastic body by use of strain gages. In order to obtain an indication of the amount of error which may result, the ratio, R_ε, of the strain $\hat{\varepsilon}_{rr}$ averaged over one radial fiber of the membrane to the uniform radial strain ε''_{rr} in the half-space was calculated. That is, we compute

$$R_\varepsilon = \frac{1}{a}\int_0^a \hat{\varepsilon}_{rr}\,\mathrm{d}r/\varepsilon''_{rr}. \tag{9.1}$$

From the strain–displacement relations with (3.5), (3.6) and (3.11), R_ε in (9.1) can be written in dimensionless form as

$$R_\varepsilon = \frac{(1-\hat{\sigma})(1+\sigma)\eta}{(1+\hat{\sigma})(1-\sigma)\gamma}\int_0^1 \frac{\zeta^2\phi(\zeta)}{(1-\zeta)^{\frac{1}{2}}}\,\mathrm{d}\zeta. \tag{9.2}$$

By use of (7.5) and the numerical solution of (7.6) the integral in (9.2) can be evaluated numerically. Figures 9(a), (b) show the dependence of R_ε on the three parameters η/γ, σ, $\hat{\sigma}$. Figure 9(a) shows how the "average membrane strain" approaches the far-field half-space strain as η/γ increases for fixed σ and $\hat{\sigma}$. In Fig. 9(b), η/γ is fixed and σ and $\hat{\sigma}$ vary. The

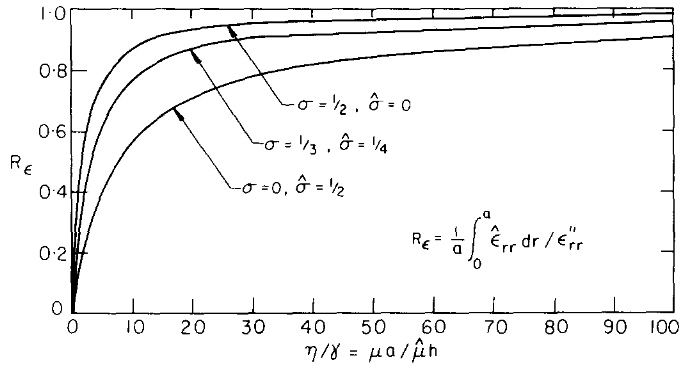

FIG. 9(a). Dependence of the strain ratio R_ε on η/γ for various $\sigma, \hat{\sigma}$.

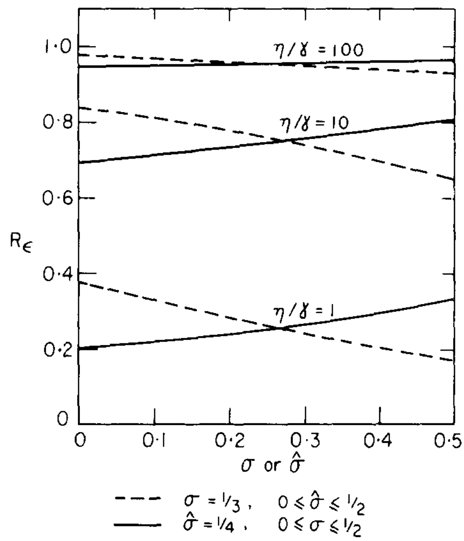

FIG. 9(b). Dependence of the strain ratio R_ε on σ (or $\hat{\sigma}$) for various $\eta/\gamma = \mu a/\hat{\mu} h$, with $\hat{\sigma} = \frac{1}{4}$ (or $\sigma = \frac{1}{3}$).

variation of R_ε is slight when η/γ is fixed, but R_ε depends strongly on η/γ in a range that depends on σ and $\hat{\sigma}$. Clearly R_ε is nearest to unity when η/γ is as large as possible and when $\sigma = \frac{1}{2}$ and $\hat{\sigma} = 0$. For example if $\eta/\gamma = 100$, $\sigma = \frac{1}{2}$ and $\hat{\sigma} = 0$, the value of R_ε is 0·980.

10. DISCUSSION AND CONCLUSIONS

In the formulation of the problem treated here, the possibility of concentrated line load transfer between the membrane and the half-space was admitted as is evident by the appearance of p, p_r and p_θ in (3.5), (3.8), (3.16) and (3.19). These functions were found to be zero by a systematic analysis of the integral equations governing the load transfer function.

The load transfer functions were found to have square root singularities at the edge of the membrane for both an inextensible and an elastic membrane.

The integral equation determining the load transfer function, which is a Fredholm equation of the first kind with a logarithmic singularity, was handled successfully by a direct numerical technique, after the known singularity in the load transfer function was removed. This equation was also solved exactly for the inextensible membrane.

The load transfer function as well as the stress fields in the membrane and half-space were found to be functions of the three parameters $\hat{\sigma}$, σ and $\mu a/\hat{\mu}h$. The variation from the far-field uniform stress in the vicinity of the attached membrane as well as the stress in the membrane was found to increase with increasing $\hat{\sigma}$ and decreasing σ and $\mu a/\hat{\mu}h$. This variation in the half-space is negligible at points more distant than three or four membrane radii from the center of the membrane.

The "average membrane strain" approximates the far-field half-space strain roughly to within 5–15 per cent, depending on σ, $\hat{\sigma}$, for effective stiffness ratios $\mu a/\hat{\mu}h$ greater than about 50.

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APPENDIX

From Ref. [20] and additional calculations there follows for $z \neq 0$

$$H_0^0 = -\frac{z}{a}\tan^{-1}\left[\frac{\rho\sin\theta + R^{\frac{1}{2}}\sin\varphi/2}{\rho\cos\theta + R^{\frac{1}{2}}\cos\varphi/2}\right] + R^{\frac{1}{2}}\sin\varphi/2,$$

$$H_1^0 = \tan^{-1}\left[\frac{\rho\sin\theta + R^{\frac{1}{2}}\sin\varphi/2}{\rho\cos\theta + R^{\frac{1}{2}}\cos\varphi/2}\right] - R^{-\frac{1}{2}}\cos\varphi/2,$$

$$H_2^0 = R^{-\frac{1}{2}}\sin\varphi/2 - \rho R^{-\frac{3}{2}}\cos(3\varphi/2-\theta),$$

$$H_0^1 = \frac{1}{2}\frac{z/a}{r/a}R^{\frac{1}{2}}\sin\varphi/2 + \frac{r/a}{2}\tan^{-1}\left[\frac{\rho\sin\theta + R^{\frac{1}{2}}\sin\varphi/2}{\rho\cos\theta + R^{\frac{1}{2}}\cos\varphi/2}\right] - \frac{1}{2}\frac{1}{r/a}R^{\frac{1}{2}}\cos\varphi/2,$$

$$H_1^1 = \frac{1}{r/a}\{\rho R^{-\frac{1}{2}}\cos(\theta-\varphi/2) - R^{\frac{1}{2}}\sin\varphi/2\},$$

$$H_2^1 = \frac{\rho}{r/a}R^{-\frac{1}{2}}\sin(\theta-\varphi/2) - \frac{r}{a}R^{-\frac{3}{2}}\cos 3\varphi/2,$$

$$H_0^2 = \frac{2}{(r/a)^2}\left\{\frac{R^{\frac{3}{2}}}{3}\sin 3\varphi/2 - \rho R^{\frac{1}{2}}\cos(\theta+\varphi/2)\right\},$$

$$H_1^2 = -\frac{1}{(r/a)^2}\left\{R^{\frac{1}{2}}\cos\varphi/2 - \frac{z}{a}R^{\frac{1}{2}}\sin\varphi/2\right\} + R^{-\frac{1}{2}}\cos\varphi/2,$$

$$H_2^2 = \frac{2}{(r/a)^2}\{\rho R^{-\frac{1}{2}}\cos(\theta-\varphi/2) - R^{\frac{1}{2}}\sin\varphi/2\} - H_2^0,$$

$$H_0^3 = \frac{2}{(r/a)^3}\left\{\frac{\rho R^{\frac{3}{2}}}{3}\sin(\theta+3\varphi/2) - \rho^2 R^{\frac{1}{2}}\cos(2\theta+\varphi/2)\right\},$$

$$H_1^3 = \frac{4}{r/a}H_0^2 - H_1^1, \qquad H_2^3 = \frac{4}{r/a}H_1^2 - H_2^1,$$

where

$$\rho^2 = 1 + \left(\frac{z}{a}\right)^2, \qquad R^2 = \left[\left(\frac{r}{a}\right)^2 + \left(\frac{z}{a}\right)^2 - 1\right]^2 + 4\left(\frac{z}{a}\right)^2$$

$$\frac{z}{a}\tan\theta = 1, \qquad \left[\left(\frac{r}{a}\right)^2 + \left(\frac{z}{a}\right)^2 - 1\right]\tan\varphi = 2\frac{z}{a},$$

and for $z = 0$

$$H_0^0 = H(1-r/a)[1-(r/a)^2]^{\frac{1}{2}},$$

$$H_1^0 = H(1-r/a)\frac{\pi}{2} + H(r/a-1)\left\{\frac{-1}{[(r/a)^2-1]^{\frac{1}{2}}} + \sin^{-1}\left(\frac{a}{r}\right)\right\},$$

$$H_0^1 = H(1-r/a)\frac{\pi}{4}\frac{r}{a} + H(r/a-1)\frac{1}{2}\left\{\frac{r}{a}\sin^{-1}\left(\frac{a}{r}\right) - \frac{1}{r/a}[(r/a)^2-1]^{\frac{1}{2}}\right\},$$

$$H_1^1 = H(1-r/a)\frac{r/a}{[1-(r/a)^2]^{\frac{1}{2}}},$$

$$H_0^2 = H(1-r/a)\frac{2}{(r/a)^2}\left\{\frac{[1-(r/a)^2]^{\frac{3}{2}}}{3}-[1-(r/a)^2]^{\frac{1}{2}}\right\},$$

$$H_1^2 = H(r/a-1)\frac{1}{(r/a)^2[(r/a)^2-1]^{\frac{1}{2}}},$$

$$H_0^3 = H(r/a-1)\frac{2}{(r/a)^3}[(r/a)^2-1]^{\frac{1}{2}}$$

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Абстракт—Дается решение задачи передачи упругостатической нагрузки в круглой упругой мембране, связанной на границе с полупространством, из разнородного материала. Полупространство нагруженное, вдали от мембраны, состоянием плоского напряжения, параллельным к его границе.

Задача сводится к системе интегральных уравнений фреугольма первого рода с логарифмическими сингулярностями в ядрах, для неизвестных сил на границе между мембраной и полупростраством. Интегральные уравнения решаются точно. Получается решение задачи в форме злементарных функции для ограниченного случая нерастяжимаемой мембраны. Решаются интегральные уравнения для упругой мембраны непосредственно путем численных методов. Получается возмущение однородного поля напряжений для присоединенной мембраны, при некоторых комбинациах параметров материала и геометрии.

Определяется некоторое отношение деформации мембраны и полупространства, которое указывает точность измерений деформации, полученных из зкспериментального анализа напряжений, путем использования присоединеннвых датчиков. Зти результаты указывают, что „усредненная деформация" в мембране близка к полю деформеции в полупространстве только тогда, когда $\mu a/\hat{\mu}h > 100$, яэе $\hat{\mu}$, h, a—являются модулем, толщиной и радиусом мембраны а μ—модулем сдвига полупространства.